Algebraic decision problems inferred from the analysis of delay-differential algebraic systems

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Overview

- Statement of decision problem
- Motivation
- A multi-dimensional Cayley-Hamilton theorem
- Solution to the decision problem
- A strengthened criterion

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Context

Finitely generated semigroup of matrices: set of all possible products of matrices taken from some finite set $\mathcal{M}$

Example

$$\mathcal{M} = \{A_1,\ A_2\} \ \Rightarrow \{A_1,\ A_2,\ A_1^2,\ A_2^2,\ A_1A_2,\ A_2A_1,\ A_1^3, A_1^2A_2,\ A_1A_2A_1,\ A_2A_1^2,\ A_1A_2^2,\ A_2A_1A_2, A_2^2A_1, A_2^3, \ldots\}$$

Application with $\mathcal{M} = \{A_1,\ A_2, \ldots, A_m\}$: switched linear system

$$x(k+1) = A_{\sigma(k)}\, x(k), \quad \sigma(k) \in \{1, \ldots, m\}.$$

Simply looking questions on semigroups of matrices often turn out to be very difficult

- is the zero matrix included in the semigroup of matrices (mortality problem)?
- does the switched system have bounded trajectories whatever the switching sequence?
- growth rate of solutions and stability?

Problem under consideration

Given a set of generators $\mathcal{M} = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n\times n}$, matrices $B \in \mathbb{R}^{n_b \times n}$ and $C \in \mathbb{R}^{n_c\times n}$, construct matrix polynomials

$$\begin{aligned} P_{k_1,\ldots,k_m}(A_1,\ldots,A_m) \quad &:= \quad A_1 P_{k_1-1,k_2,\ldots,k_m}(A_1,\ldots,A_m) + A_2 P_{k_1,k_2-1,\ldots,k_m}(A_1,\ldots,A_m) \\ & \qquad + \cdots + A_m P_{k_1,k_2,\ldots,k_m-1}(A_1,\ldots,A_m), \qquad k_j \in \mathbb{Z}_{\geq 0},\ j = 1,\ldots,m \end{aligned}$$

$$P_{0,\ldots,0}(A_1,\ldots,A_m) := I$$

$$P_{k_1,\ldots,k_m}(A_1,\ldots,A_m) := 0 \text{ if any } k_j \in \mathbb{Z}_{<0},\ j = \ 1,\ldots,m$$

Decision problem

Find a finite test for determining that the following conditions hold:

$$CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B = 0,\ \forall (k_1,\ldots,k_m) \in \mathbb{Z}^m_{\geq 0}$$

For $m = 2,\ \mathcal{M} = \{A_1,\ A_2\}$:

$$\begin{aligned} CP_{0,0}B &= CB = 0, \\ CP_{1,0}B &= CA_1B = 0, & CP_{0,1}B &= CA_2B = 0, \\ CP_{2,0}B &= CA_1^2B = 0, & CP_{1,1}B &= C\left(A_1A_2 + A_2A_1\right)B = 0, \\ CP_{0,2}B &= CA_2^2B = 0, & CP_{3,0}B &= CA_1^3B = 0, \\ CP_{2,1}B &= C(A_1^2A_2 + A_1A_2A_1 + A_2A_1^2)B = 0,\ \ldots \end{aligned}$$

Case $m = 1$, $\mathcal{M} = \{A\}$ $\rightarrow$ Find a finite test for determining that $CA^kB = 0,\ \forall k \in \mathbb{Z}_{\geq 0}$

Sufficient to test for $k = 0, \ldots, n-1$ by the Cayley Hamilton (CH) theorem

$$p(x) := \det(xI - A) \ \rightarrow\ p(A) = 0 \ \rightarrow A^n = \cdots .$$

Not sufficient to test "component-wise" for $m > 1$

Consider $m = 2$ and matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},\quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

By direct calculation:

$$CA_1^kB = 0,\ \text{ and } CA_2^kB = 0,\quad k = 0, 1, 2$$

but

$$C\,(A_1A_2 + A_2A_1)\,B \neq 0.$$

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Delay differential algebraic systems

$$E\frac{d}{dt}\tilde{x}(t)=\sum_{j=0}^{m}\tilde{A}_j\tilde{x}(t-h_j)+\tilde{B}u(t),\quad E \text{ possibly singular},\quad 0=h_0<h_1<\cdots<h_m$$

$$\tilde{y}(t)=\tilde{C}x(t)$$

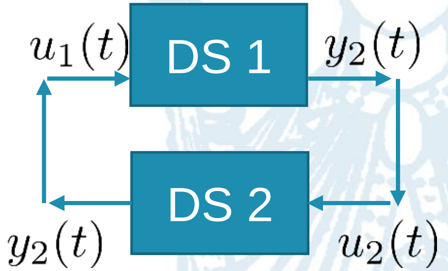
$$\Downarrow$$

$$\text{e.g. } E=\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}$$

Standard form, at the basis of several algorithms and software

- natural in modeling interconnected systems

$$I(t)=C\frac{dV(t)}{dt}$$
$$V(t)=L\frac{dI(t)}{dt}$$
$$V(t)=RI(t)$$
$$\sum I_k=0$$
$$\sum V_\ell=0$$



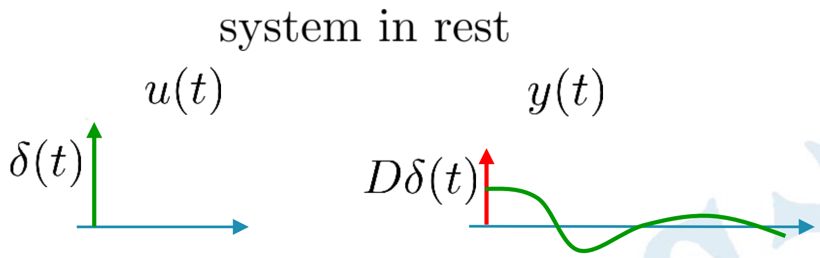
$$\Downarrow$$

- closed-loop system: no elimination of inputs and outputs needed
- allows to model systems with input / output delays

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & Ax(t)+B_1u(t)+B_2u(t-h)\\ y(t) & = & Cx(t)\end{array}\right.$$

$$\Downarrow$$

$$\left\{\begin{array}{rcl}\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\dot{X}(t) & = & \begin{pmatrix} A & B_1\\ 0 & -I\end{pmatrix}X(t)+\begin{pmatrix} 0 & B_2\\ 0 & 0\end{pmatrix}X(t-h)+\begin{pmatrix} 0\\ I\end{pmatrix}u(t)\\ y(t) & = & (C\ \ 0)\,X(t)\end{array}\right.$$



$$G(s) = F\left(sI - A_1 - A_2 e^{-sh}\right)^{-1} B + D$$

- allows to model systems with a nontrivial feed-through

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & A_1 x(t) + A_2 x(t-h) + Bu(t) \\ y(t) & = & Fx(t) + Du(t) \end{array}\right.$$

$$\Downarrow$$

$$\left\{\begin{array}{l} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \dot{X}(t) = \begin{pmatrix} A_1 & 0 \\ 0 & -I \end{pmatrix} X(t) + \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix} X(t-h) + \begin{pmatrix} B \\ I \end{pmatrix} u(t) \\ y(t) = (F \;\; D)\, X(t) \end{array}\right.$$

- allows to model time-delay systems of neutral type

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & A_1 x(t) + A_2 x(t-h) + A_3 \dot{x}(t-h) + Bu(t) \\ y(t) & = & Fx(t) \end{array}\right.$$

$$\Downarrow \qquad \frac{d}{dt}\left(x(t) - A_3 x(t-h)\right) = A_1 x(t) + A_2 x(t-h)$$

$$\left\{\begin{array}{l} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \dot{X}(t) = \begin{pmatrix} A_1 & 0 \\ I & -I \end{pmatrix} X(t) + \begin{pmatrix} A_2 & 0 \\ -A_3 & 0 \end{pmatrix} X(t-h) + \begin{pmatrix} B \\ 0 \end{pmatrix} u(t) \\ y(t) = (F \;\; 0)\, X(t) \end{array}\right.$$

H2 norm analysis

$$E\frac{d}{dt}\tilde{x}(t)=\sum_{j=0}^{m}\tilde{A}_j\tilde{x}(t-h_j)+\tilde{B}u(t),\quad E \text{ possibly singular},\quad 0=h_0<h_1<\cdots<h_m$$

$$\tilde{y}(t)=\tilde{C}x(t)$$

Assumptions:

1. $U^T\tilde{A}_0V$ is invertible, where the columns of U, respectively V form a basis of the left, respectively right null space of E
2. for $u\equiv 0$, the zero solution is exponentially stable

$$\|G\|_{\mathcal{H}_2}:=\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Tr}(G^*(i\omega)G(i\omega))d\omega}=\sqrt{\int_0^{\infty}\mathrm{Tr}(h^T(t)h(t))dt}$$

$$\delta(t)\longrightarrow \boxed{\text{DS}} \longrightarrow h(t)$$

$$x_0=0$$

$h(t)$: impulse response $\xrightarrow{\text{Laplace}}$ $G(s)=\tilde{C}\left(sE-\sum_{j=0}^{m}\tilde{A}_je^{-sh_j}\right)^{-1}\tilde{B}$: transfer function

Under Assumptions 1-2 the H2 norm can still be not finite

Finiteness of H2 norm

$$E\frac{d}{dt}\tilde{x}(t) = \sum_{j=0}^{m} \tilde{A}_j \tilde{x}(t-h_j) + \tilde{B}u(t)$$
$$\tilde{y}(t) = \tilde{C}x(t)$$

$$G(s) = \tilde{C}\left(sE - \sum_{j=0}^{m} \tilde{A}_j e^{-sh_j}\right)^{-1} \tilde{B}$$

Under Assumption 1:
transformation $\tilde{x}(t) \leftrightarrow \begin{pmatrix} x_1(t) \\ x(t) \end{pmatrix}$

Asymptotic behavior on imaginary axis

$$\frac{d}{dt}x_1(t) = \sum_{j=0}^{m} A_j^{(11)} x_1(t-h_j) + \sum_{j=0}^{m} A_j^{(12)} x(t-h_j) + B_1 u(t)$$
$$x(t) = \sum_{j=0}^{m} A_j^{(21)} x_1(t-h_j) + \sum_{j=1}^{m} A_j x(t-h_j) + Bu(t)$$
$$y(t) = C_1 x_1(t) + Cx(t)$$

$$G_a(s) = C(I + \sum_{j=1}^{m} A_j e^{-sh_j})^{-1} B$$

impulse response h_a

Theorem:
Under Assumptions 1-2, the H2 norm of G is finite if and only if G_a is identically zero.

SEMI-INFINITE EQUALITY

Reduced problem

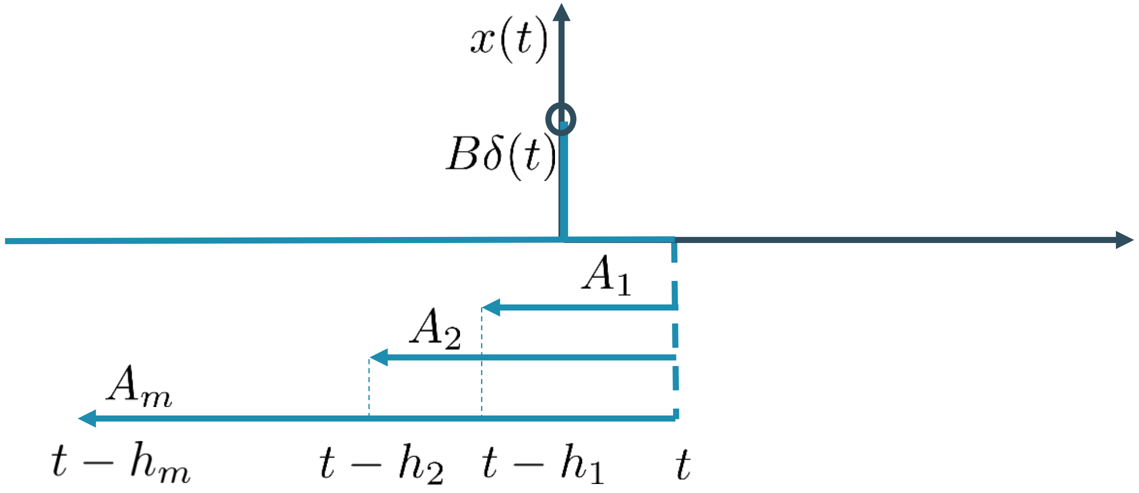
Delay-difference equation

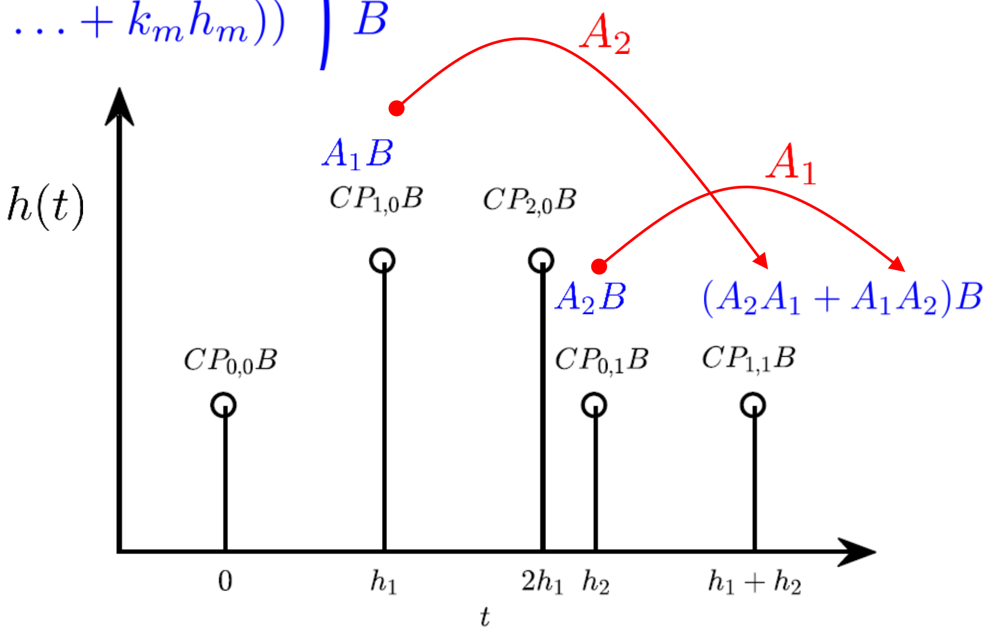
$$x(t) = \sum_{j=1}^{m} A_j x(t-h_j) + Bu(t), \qquad 0 < h_1 < \cdots < h_m$$

$$y(t) = Cx(t)$$

Impulse response $h(t)$ $\leftrightarrow$ Transfer function $G_a(s) := C\left(I - \sum_{j=1}^{m} A_j e^{-sh_j}\right)^{-1} B$

$$h(t) = C\left(\sum_{k_1,\dots,k_m=0}^{\infty} P_{k_1,\dots,k_m}(A_1,\dots,A_m) \cdot \delta\left(t-(k_1h_1+\dots+k_mh_m)\right)\right)B$$





$$x(t)=\sum_{j=1}^{m}A_jx(t-h_j)+Bu(t),\qquad y(t)=Cx(t)$$

$$h(t)=\sum_{k_1,\ldots,k_m=0}^{\infty}\left(C\;P_{k_1,\ldots,k_m}(A_1,\ldots,A_m)\;B\right)\;\cdot\;\delta\left(t-(k_1h_1+\ldots+k_mh_m)\right)$$

When is h identically zero?

Case 1: delays $(h_1,\ldots,h_m)$ are *rationally independent* $\longrightarrow$

$(\sum_{i=1}^{m} z_ih_i=0, z_i\in\mathbb{Z},\ i=1,\ldots,m \text{ implies } z_i=0,\ i=1,\ldots,m)$

Not possible that for any m-tuples $(k_1,\ldots,k_m)\in\mathbb{Z}^m_{\geq 0}$ and $(k_1^*,\ldots,k_m^*)\in\mathbb{Z}^m_{\geq 0}$ such that $(k_1,\ldots,k_m)\neq(k_1^\star,\ldots,k_m^\star)$ and

$$k_1h_1+\ldots+k_mh_m=k_1^*h_1+\ldots+k_m^*h_m$$

If delays $(h_1,\ldots,h_m)$ are rationally independent, then condition

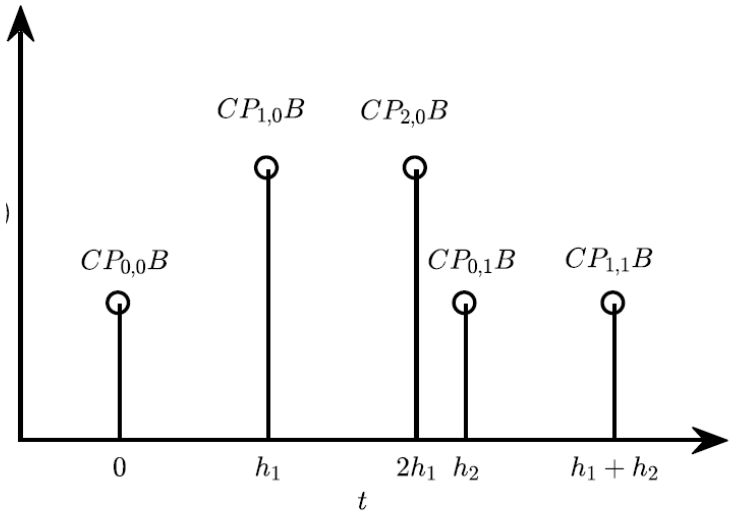
$$CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B=0,\ \forall(k_1,\ldots,k_m)\in\mathbb{Z}^m_{\geq 0}$$

is necessary and sufficient for $h\equiv 0$ or $G_a\equiv 0$.

INFINITE BUT *COUNTABLE*
NUMBER OF EQUALITIES

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j) + Bu(t), \qquad y(t) = Cx(t)$$

$$h(t) = \sum_{k_1,\ldots,k_m=0}^{\infty} \left(C\ P_{k_1,\ldots,k_m}(A_1,\ldots,A_m)\ B\right) \cdot \delta\left(t - (k_1 h_1 + \ldots + k_m h_m)\right)$$



Case 2: delays $(h_1, \ldots, h_m)$ are *rationally dependent*

The condition on previous slide is sufficient but not necessary.
However

- Set of rationally independent delays is dense in $\mathbb{R}^m_{\geq 0}$
- There always exist a smaller number p of rationally independent delays $(r_1, \ldots, r_p)$ and a matrix $R \in \mathbb{Z}^{m\times p}_{\geq 0}$ of full column rank such that

$$\begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = R \begin{pmatrix} r_1 \\ \vdots \\ r_p \end{pmatrix}.$$

Transformation to a problem with rationally independent delays always possible

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However

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$$\begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = R \begin{pmatrix} r_1 \\ \vdots \\ r_p \end{pmatrix}.$$

Transformation to a problem with rationally independent delays always possible

$$\begin{aligned} x(t) &= A_1 x(t-r_1) + A_2 x(t-r_2) + A_3 x(t-r_1-r_2) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \begin{pmatrix} x(t) \\ x_2(t) \end{pmatrix} &= \begin{pmatrix} A_1 & A_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t-r_1) \\ x_2(t-r_1) \end{pmatrix} + \begin{pmatrix} A_2 & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} x(t-r_2) \\ x_2(t-r_2) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_2(t) \end{pmatrix} \end{aligned}$$

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Concept

- *Recurrence relation* between multi-variate powers of a (block) matrix
- Generalization of results of Givone and Roessler (IEEE Trans. Computers, 1973) and B. Vilfan (IEEE Trans. Computers, 1973)
- Grounded in the standard CH Theorem

$$D = \begin{pmatrix} D_{11} & \dots & D_{1m} \\ \vdots & & \vdots \\ D_{m1} & \dots & D_{mm} \end{pmatrix} \in \mathbb{R}^{mn \times mn}$$

$$D^{[1,0,\dots,0]} := \begin{pmatrix} D_{11} & \dots & D_{1m} \\ & 0 & \\ & 0 & \end{pmatrix}, \quad \dots, \quad D^{[0,\dots,0,1]} := \begin{pmatrix} & 0 & \\ & 0 & \\ D_{m1} & \dots & D_{mm} \end{pmatrix}.$$

$$D^{[k_1,\dots,k_m]} := D^{[1,\dots,0]} D^{[k_1-1,\dots,k_m]} + \dots + D^{[0,\dots,1]} D^{[k_1,\dots,k_m-1]}, \qquad (k_1,\dots,k_m) \in \mathbb{Z}^m_{\geq 0}$$

$$D^{[k_1,\dots,k_m]} := 0 \text{ if any } k_j \in \mathbb{Z}_{<0}$$

Main result

$$D = \begin{pmatrix} D_{11} & \dots & D_{1m} \\ \vdots & & \vdots \\ D_{m1} & \dots & D_{mm} \end{pmatrix} \in \mathbb{R}^{mn \times mn}$$

Let

$$f(x_1, \dots, x_m) := \det\left(\begin{pmatrix} x_1 I_n & \dots & 0 \\ & \ddots & \\ 0 & \dots & x_m I_n \end{pmatrix} - D \right) = \sum_{k_1, \dots, k_m = 0}^{n} a_{k_1, \dots, k_m} x_1^{k_1} \dots x_m^{k_m}.$$

Then expression

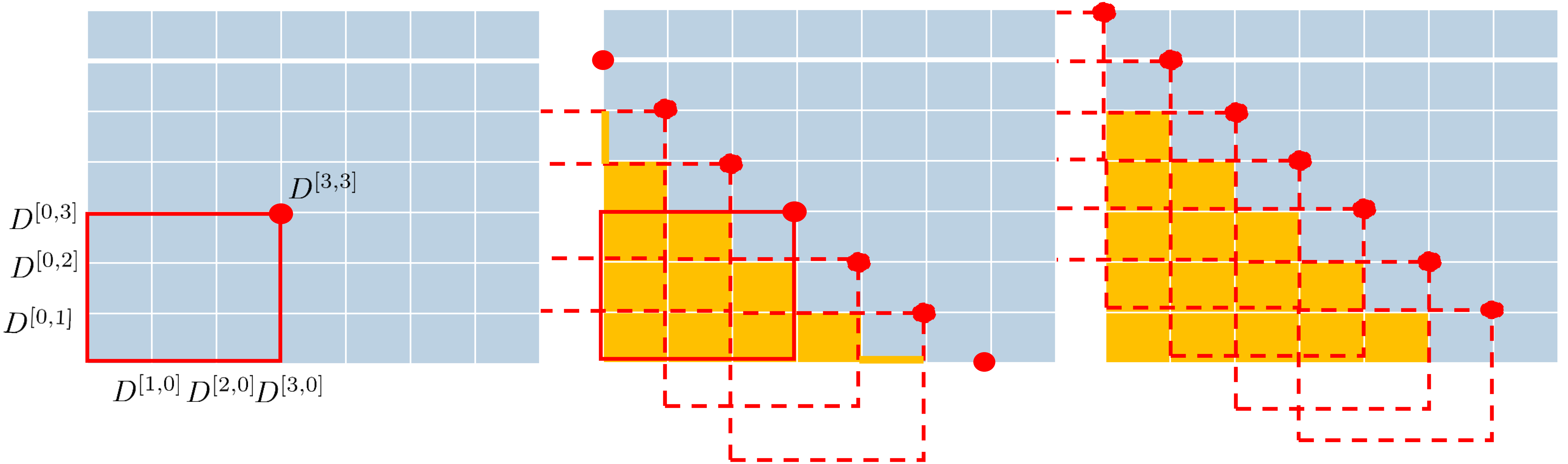
$$D^{[n+p_1, \dots, n+p_m]} = - \sum_{\substack{k_1, \dots, k_m = 0 \\ k \neq (n, \dots, n)}}^{n} a_{k_1, \dots, k_m} D^{[k_1+p_1, \dots, k_m+p_m]}$$

holds for any $(p_1, \dots, p_m) \in \mathbb{Z}^m$ satisfying $\sum_{i=1}^{m} p_i \geqslant 0$.

$$D^{[n+p_1,\ldots,n+p_m]} = -\sum_{\substack{k_1,\ldots,k_m=0 \\ k\neq(n,\ldots,n)}}^{n} a_{k_1,\ldots,k_m} D^{[k_1+p_1,\ldots,k_m+p_m]}, \quad \text{holds for any } (p_1,\ldots,p_m)\in\mathbb{Z}^m \text{ satisfying } \sum_{i=1}^{m} p_i \geqslant 0.$$

Example with $(m,n)=(2,3)$:

$p_1 = p_2 = 0$ $p_1 + p_2 = 0$ $p_1 + p_2 = 1$



Minimal information to generate all powers with recursion formula?

It is sufficient to know $D^{[k_1,\ldots,k_m]}$, for $k_1+\ldots+k_m < mn$! All other powers **are linear combinations**!

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Sketch of the proof

$$f(x_1,\dots,x_m) := \det\left(\begin{pmatrix} x_1 I & \dots & 0 \\ & \ddots & \\ 0 & \dots & x_m I \end{pmatrix} - D\right) = \sum_{k_1,\dots,k_m=0}^{n} a_{k_1,\dots,k_m} x_1^{k_1}\dots x_m^{k_m}$$

$$g(x) := \det(xI - D) = \sum_{j=0}^{mn} a_j x^j$$

$$\left.\begin{matrix} f(x_1,\dots,x_m) = \dots \\ g(x) = \dots \end{matrix}\right\} \quad g(x) = f(x,\dots,x) \Rightarrow$$

$$a_j = \sum_{k\in\Omega_j} a_{k_1,\dots,k_m}, \quad \text{with}$$

$$\Omega_j := \{k \in \mathbb{Z}^m : \sum_{i=1}^{m} k_i = j\}$$

From definition of multi-variate powers: $D^j = \sum_{l\in\Omega_j} D^{[l_1,\dots,l_m]}$

Standard Cayley-Hamilton theorem: $\sum_{j=0}^{mn} a_j D^j = 0 \Rightarrow \sum_{j=0}^{mn} \sum_{k\in\Omega_j} \sum_{l\in\Omega_j} a_{k_1,\dots,k_m} D^{[l_1,\dots,l_m]} = 0$

Sketch of proof (II)

$$\sum_{j=0}^{mn} \sum_{k \in \Omega_j} \sum_{l \in \Omega_j} a_{k_1,\ldots,k_m} D^{[l_1,\ldots,l_m]} = 0 \qquad D = \begin{pmatrix} D_{11} & \ldots & D_{1m} \\ \vdots & & \vdots \\ D_{m1} & \ldots & D_{mm} \end{pmatrix}$$

degree l_i in elements of the i-the block row of D

degree $n - k_i$ in elements of the i-the block row of D

- Restriction to terms of degree n in elements of all block rows: $l_i = k_i, \;\; i = 1, \ldots, m$

$$\sum_{j=0}^{mn} \sum_{k \in \Omega_j} a_{k_1,\ldots,k_m} D^{[k_1,\ldots,k_m]} = 0 : \quad \text{recursion formula for } p_1 = \cdots = p_m = 0$$

- Restriction to terms of degree $n + p_i$ in elements of i-th block row, with $\sum p_i = 0$:
  $l_i = k_i + p_i, \;\; i = 1, \ldots, m$

$$\sum_{j=0}^{mn} \sum_{k \in \Omega_j} a_{k_1,\ldots,k_m} D^{[k_1+p_1,\ldots,k_m+p_m]} = 0 : \quad \text{recursion formula for } \sum p_i = 0$$

- Recursive definition of $D^{[k_1,\ldots,k_m]}$: recursion formulae for $\sum p_i > 0$

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A finite test for the finiteness of the H2 norm

$$x(t) = \sum_{j=1}^{m} A_j x(t-h_j) + Bu(t)$$
$$y(t) = Cx(t)$$

Impulse response h_a, transfer function G_a

If the delays $(h_1, \ldots, h_m)$ are rationally independent, then condition

$$CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B = 0, \quad \text{for all } (k_1,\ldots,k_m) \in \mathbb{Z}^m_{\geq 0}$$

is necessary and sufficient for $h \equiv 0$ or $G_a \equiv 0$

$$D = \begin{pmatrix} A_1 & \ldots & A_1 \\ A_2 & \ldots & A_2 \\ \vdots & & \vdots \\ A_m & \ldots & A_m \end{pmatrix} \Rightarrow \quad P_{k_1,\ldots,k_m}(A_1,\ldots,A_m) = \frac{1}{m}\left(I_n \cdots I_n\right) A^{[k_1,\ldots,k_m]} \begin{pmatrix} I_n \\ \vdots \\ I_n \end{pmatrix}$$

Example: $m = 2 \;\rightarrow\; D^{[1,1]} = \begin{bmatrix} A_1 A_2 & A_1 A_2 \\ A_2 A_1 & A_2 A_1 \end{bmatrix}$

If the delays $(h_1, \ldots, h_m)$ are rationally independent, then condition

$CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B = 0$, for all $(k_1,\ldots,k_m) \in \mathbb{Z}^m_{\geq 0}$ such that $k_1 + \ldots + k_m < mn$

is necessary and sufficient for $h \equiv 0$ or $G_a \equiv 0$

FINITE
NUMBER OF EQUALITIES

Finiteness criterion of H2 norm

$$E\frac{d}{dt}\tilde{x}(t) = \sum_{j=0}^{m} \tilde{A}_j \tilde{x}(t-h_j) + \tilde{B}u(t)$$

$$\tilde{y}(t) = \tilde{C}x(t)$$

$$G(s) = \tilde{C}\left(sE - \sum_{j=0}^{m} \tilde{A}_j e^{-sh_j}\right)^{-1}\tilde{B}$$

Assumptions:
index 1, exponentially stable
rationally independent delay

Asymptotic behavior
on imaginary axis

$$\frac{d}{dt}x_1(t) = \sum_{j=0}^{m} A_j^{(11)} x_1(t-h_j) + \sum_{j=0}^{m} A_j^{(12)} x(t-h_j) + B_1 u(t)$$

$$x(t) = \sum_{j=0}^{m} A_j^{(21)} x_1(t-h_j) + \sum_{j=1}^{m} A_j x(t-h_j) + Bu(t)$$

$$y(t) = C_1 x_1(t) + Cx(t)$$

$$G_a(s) = C(I + \sum_{j=1}^{m} A_j e^{-sh_j})^{-1} B$$

Necessary and sufficient condition
$CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B = 0$, for all $(k_1,\ldots,k_m) \in \mathbb{Z}^m_{\geq 0}$ such that $k_1 + \ldots + k_m < mn$

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A sufficient finiteness condition

Necessary and sufficient condition
$CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B = 0$, for all $(k_1,\ldots,k_m) \in \mathbb{Z}^m_{\geq 0}$ such that $k_1 + \ldots + k_m < mn$

- Still computationally expensive

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$
$n=4$	36	364	3876	42504	475020	5379616	61523748	708930508
$n=5$	55	680	8855	118755	1623160	22481940	314457495	4431613550
$n=6$	78	1140	17550	278256	4496388	73629072	1217566350	20286591270
$n=7$	105	1771	31465	575757	10737573	202927725	3872894697	74473879480

Number of equalities to check

- How to compute the H2 norm, when it is finite?

Sufficient finiteness condition

$$\begin{aligned} CB &= 0, \\ CA_{\sigma_1}\cdots A_{\sigma_k}B &= 0, \;\; \forall k \in \mathbb{Z}_{>0},\; \forall \sigma_i \in \{1,\ldots,m\},\; i = 1,\ldots,k \end{aligned}$$

“C x (any Monomial) x B=0”

Properties of new criterion

Sufficient condition

$$\begin{aligned} CB &= 0, \\ CA_{\sigma_1} \cdots A_{\sigma_k} B &= 0, \ \forall k \in \mathbb{Z}_{>0}, \ \forall \sigma_i \in \{1, \ldots, m\}, \ i = 1, \ldots, k \end{aligned}$$

Property 1: efficient computational procedure

$$\begin{aligned} &\chi_0 := \text{span } B \\ &\chi_k := \text{span } \{B\} \cup \left\{A_{\sigma_1} \cdots A_{\sigma_{k'}} B : \ 1 \leq k' \leq k, \ \sigma_i \in \{1, \ldots, m\}, \ i = 1, \ldots, k'\right\}, \end{aligned}$$

$\Downarrow$

$$\chi_{k+1} = \chi_k \cup \{A_j \chi_k : j = 1, \ldots, m\}$$

$\Downarrow$

Sufficient conditions holds $\Leftrightarrow$ for some k, χ_k is an $(A_1, \ldots, A_m)$-invariant set in Ker C.

$\Downarrow$

$\Downarrow$

Algorithm based on computing less or equal to $m(n - \text{rank } C)$ matrix vector products with A_i

Property 2: induces a transformation to standard neutral functional differential equation

$$
\begin{aligned}
\frac{d}{dt}x_1(t) &= \sum_{j=0}^{m} A_j^{(11)} x_1(t-h_j) + \sum_{j=0}^{m} A_j^{(12)} x(t-h_j) + B_1 u(t) \\
x(t) &= \sum_{j=0}^{m} A_j^{(21)} x_1(t-h_j) + \sum_{j=1}^{m} A_j x(t-h_j) + Bu(t) \\
y(t) &= C_1 x_1(t) + Cx(t)
\end{aligned}
$$

Sufficient finiteness condition is equivalent to the existance of an invertible matrix T_c such that

$$
T_c^{-1} A_j T_c = \left[\begin{array}{c|c} A_{j1} & 0 \\ \hline A_{j2} & A_{j3} \end{array}\right],\ j = 1, \ldots, m,\ \ T_c^{-1} B = \left[\begin{array}{c} 0 \\ \hline B_c \end{array}\right],\ CT_c = \left[\begin{array}{c|c} C_u & 0 \end{array}\right]
$$

$$
\overbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}^{\dot{x}(t)} = \quad \overbrace{\begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}}^{x(t)} \quad \overbrace{\begin{bmatrix} A_{j1} & 0 \\ A_{j2} & A_{j3} \end{bmatrix}}^{x(t-h_j)} \quad \overbrace{\begin{bmatrix} 0 \\ B_c \end{bmatrix}}^{u(t)}
$$

$$
y(t) \left\{ \begin{bmatrix} C_u & 0 \end{bmatrix} \right.
$$

$$\overbrace{\begin{bmatrix}0&0\\0&0\end{bmatrix}}^{\dot{x}(t)} = \overbrace{\begin{bmatrix}-I&0\\0&-I\end{bmatrix}}^{x(t)} \quad \overbrace{\begin{bmatrix}A_{j1}&0\\A_{j2}&A_{j3}\end{bmatrix}}^{x(t-\tau_j)} \quad \overbrace{\begin{bmatrix}0\\B_c\end{bmatrix}}^{u(t)}$$

$$y(t)\ \{\begin{bmatrix}C_u & 0\end{bmatrix}$$

differentiate first equation

$$\overbrace{\begin{bmatrix}I&0\\0&0\end{bmatrix}}^{\dot{x}(t)} = \overbrace{\begin{bmatrix}0&0\\0&-I\end{bmatrix}}^{x(t)} \quad \overbrace{\begin{bmatrix}A_{j1}&0\\0&0\end{bmatrix}}^{\dot{x}(t-\tau_j)} \quad \overbrace{\begin{bmatrix}0&0\\A_{j2}&A_{j3}\end{bmatrix}}^{x(t-\tau_j)} \quad \overbrace{\begin{bmatrix}0\\B_c\end{bmatrix}}^{u(t)}$$

$$y(t)\ \{\begin{bmatrix}C_u & 0\end{bmatrix}$$

transposed system $\left\| C\left(I-\sum_{i=1}^{m} A_i e^{-s\tau_i}\right)^{-1} B\right\|_{\mathcal{H}_2} = \left\| B^T\left(I-\sum_{i=1}^{m} A_i^T e^{-s\tau_i}\right)^{-1} C^T\right\|_{\mathcal{H}_2}$

$$\begin{bmatrix}I&0\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-I\end{bmatrix} \quad \begin{bmatrix}A_{j1}^T&0\\0&0\end{bmatrix} \quad \begin{bmatrix}0&A_{j2}^T\\0&A_{j3}^T\end{bmatrix} \quad \begin{bmatrix}C_c^T\\0\end{bmatrix}$$

$$\begin{bmatrix}0 & B_u^T\end{bmatrix}$$

differentiate second equation

$$\begin{bmatrix}I&0\\0&I\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} \quad \begin{bmatrix}A_{j1}^T&0\\0&A_{j3}^T\end{bmatrix} \quad \begin{bmatrix}0&A_{j2}^T\\0&0\end{bmatrix} \quad \begin{bmatrix}C_c^T\\0\end{bmatrix}$$

$$\begin{bmatrix}0 & B_u^T\end{bmatrix}$$

Property 3: sufficient condition is not necessary

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, C^T = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

One can verify by direct calculation that

$$CP_{k_1,k_2}(A_1, A_2)B = 0$$

for every (k_1, k_2) such that $k_1 + k_2 < 8$, but

$$CA_1A_2B = -1, \quad CA_2A_1B = 1,$$

Concluding remarks

- Rich dynamics of even linear time-invariant DDAEs
- Basic question (*is H2 norm finite? / is there a direct connection between input and output?*) induced algebraic decision problems

$$CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B = 0,\ \forall (k_1,\ldots,k_m) \in \mathbb{Z}^m_{\geq 0}???$$
$$CB = 0,\ \ CA_{\sigma_1}\cdots A_{\sigma_k}B = 0,\ \ \forall k \in \mathbb{Z}_{>0},\ \forall \sigma_i \in \{1,\ldots,m\},\ \ i = 1,\ldots,k???$$

- Decision problems turned into finite tests
- Engineering applications as catalyst for mathematical questions